In scientific work we often ... value of a no.

actual	Approx.
π	3.14159
e	2.71828
$\sqrt{2}$	1.414

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In this topic we are interested in finding out approximate values of a number

Any number has digits along or before decimal and after decimal. The rightmost place is known as least significant place and left most place is most significant place

integer → fixed No.
decimal/real No. → floating No.

Most significant digit ↓ - - - - - ↓ least significant digit

e.g $\underset{MSD}{1}57.13\underset{LSD}{1}$

The number of digits essentially required to represent a real No. ignoring exponent of base 10 ~~write a number~~ is known as number of significant digits in a real number.

$M \times 10^{Exponent}$; M : Mentisa

$2.3 \rightarrow = 23 \times 10^{-1}$

$= 0.23 \times 10$

$= 0.0023 \times 10^{2}$

in all cases 2 significant digits.

Q. The number of significant digits in the following numbers are :

1) 0.0005603000
2) 23.00006
3) 002.11006000

$\Rightarrow$ $0.0005603000 = 0.5603 \times 10^{-3}$

$\therefore$ 4 significant digits.

23.00006 ← 7 significant digits

0.02.11006000 ← 6 significant digits

* If any number X is given and we want to approximate it by another number X_1 which has n-significant digits then two types of approximations are taken:

1. Truncation method.

In Truncation method, n-digits of the given numbers from left is taken and remaining are Chopped off (काटके फेंकना)

e.g. $X = 23.718928$

What will be it's truncated value of X if it is truncated by five significant digit?

$\Rightarrow X_1 = 23.718 \leftarrow$ Truncated value.

2. Round-off method.

In this method of approximation upto n-significant places we take approximate value as 1st n-places from left and remove the remaining portion.

(a) If removed portion is less than half of place value at nth place then the truncated value - is the Round-off value.

$\frac{1}{2} \times$ place value of nth place

(b) If removed portion is greater than half of it's place value at nth place then add one at the nth significant place in the taken value

(c) If removed portion is equal to half of its value at nth place then it's nth place in taken value has even digit. (add nothing).

If n^{th} place in taken value has odd digit then add '1' at the n^{th} place

Round of the following digits upto 4^{th} ~~decimal~~ significant digits

1) 41.1551002
2) 41.1549491
3) 41.155
4) 41.165

41.15|51002	→ 41.16
41.15|49491	41.15
41.15|5	41.16
41.16|5	41.16

4^{th} significant place of place value is $\frac{1}{100} = 0.01$

half of the place value $= 0.005$

~~+1~~

~~0.006~~

~~$\therefore 41.15 + 0.006 = 41.156$~~

41.15

(1) $0.0051002 > 0.005$

Thus $41.15 + 0.01 = 41.16$

(2) $0.0049491 < 0.005$ thus do nothing 41.15

(3) $0.005 = 0.005 \; \{= \frac{1}{2} \times 0.01\}$ (n^{th} significant place value)

& n^{th} place value odd thus add 1.

(4) n^{th} significant value is even thus add nothing.

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Q. If the numbers given in 3-significant digits are 24.6, 2.46 and 0.246 and there sum will be ~~reported~~ reported as ___ ?

$\Rightarrow$ World of 3-significant digits-

$$\begin{array}{r} 24.6 \\ 2.46 \\ .246 \\ \hline 27.3|06 \end{array}$$

World of 3-significant digits

By default no method is given for approximation then it is Round-off method

$\therefore$ Ans = ~~28.7~~ 27.3

Error

If X is true value of number and if X_1 is approximate value of number then absolute error in approximation

E_A = True value − approximate value

$= X - X_1$

Relative error : $E_R = \frac{E_A}{X}$

Percentage error : $E_P = E_R \times 100$

Q. If 24.5 is approximated as 24 then what is absolute error in approxiⁿ ?

What is relative error ?

What is percentage error ?

$\Rightarrow$ ~~0.5~~,

Q.

$E_A = 0.5$

$E_R = \frac{0.5}{24.5}$

$E_p = \frac{0.5}{24.5} \times 100\ \%$

\# If we have any function whose value is to be approximated at any point then in expansion of that function by taylors theorem about any point we take upto certain terms which involves finite powers of x and remaining terms are truncated thus approximation of every function in Numerical analysis is a polynomial function

Note

If ~~the~~ round ~~of~~ - off a number upto n-places after decimal then error in round-off has it's magnitude is,

$$\left| \text{Error}\big|_{\text{Round off}} \right| \leq \frac{1}{2} \times 10^{-n}$$

$$\left| (\text{Round Error}) \right| \leq \frac{1}{2} \times 10^{-n}.$$

$$\left| \text{Error}_{\text{Truncation}} \right| < 10^{-n}$$

Q. If we want to approximate the value of e^x at $x = 1$ correctly upto two places after decimal then how many terms in taylors expansion of e^x about $x = 0$ will be taken?

$\Rightarrow$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

1st n - terms

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

$$\left|\frac{x^n}{n!}\right| \le \frac{1}{2} \times 10^{-2} \quad ;$$

छुटने वाला सबसे पहला term.

remainder after n-terms approximately

is $\left|\frac{x^n}{n!}\right| \le \frac{1}{2} \times 10^{-2}$ ← 2-places तक

at $x=1$, $\left|\frac{1}{n!}\right| \le \frac{1}{200}$

$$n! \geq 200$$

$n=1$, $1!$

$n=2$ $\quad 2!$ $\qquad$ 1 2 3 4 5 6 7 8

$n=3$ $\quad 3! = 6$

$n=4$ $\quad 4! = 24$

$n=5$ $\quad 5! = 120$

$n=6$ $\quad 6! = 720$

$\therefore$ leave term from $n=6$ onwards.

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$x=1 \Rightarrow e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

	1
$/2$	0.5
$/3$	$0.166 \Leftarrow 0.5/3$
$/4$	$0.0415 \Leftarrow 0.166/4$
$/5$	$0.0083 \Leftarrow 0.0415/5$
	2.7158

Q. If we want to find approximate value of $1/\sqrt{1+x}$ for very very small value of x then which formula will be used for this purpose of approximation

$$\Rightarrow (1+x)^{-\frac{1}{2}} = 1 - (1/2)x + \frac{(-1/2)(-1/2-1)}{2!}x^2 + \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}x^3 + \dots$$

The best formula to approximate $1/\sqrt{1+x}$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

Q. If we want to find approximate value of $\sqrt{1+x}$ at $x = 1.23456789101112 \times 10^{-12}$ on a machine which takes no. of 15 digits then the formula used for its accurate approx. is

(1) $\sqrt{1+x}$ itself (2) $\frac{1+x}{\sqrt{1+x}}$

(3) $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \text{—}$ (4) None

If we are approximating polynomial of degree n then remainder will be $(n+1)^{th}$ term.

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Q. If e^x is approximate by Taylors expansion as polynomial function at $x = 1/2$ then what must be degree of polynomial so that error in approximation does not exceed 10^{-2}

⇒ Let us assume that approximation is done by polynomial of order n then

$$\text{error} = \left.\frac{x^{n+1}}{(n+1)!}\right|_{x=\frac{1}{2}} \leq \frac{1}{100}$$

$$\frac{1}{(2^{n+1})\,(n+1)!} \leq \frac{1}{100}$$

$n+1 = 4$ ⇒ $\lfloor n+1 \; 2^{n+1} \geq 100$ ⇒ $n \neq 14$

$n = 3$

$$\cancel{\frac{1}{2^4\,(4)!} \leq \frac{1}{100}}$$

∴ The degree of polynomial must be $n = 3$.

Q. If we want to evaluate $\sqrt{1+x} - 1$ at $x = 0.123456789 \times 10^{-8}$ then which formula will be taken to find it's approximate value with much accuracy?

⇒ (1) $\sqrt{1+x} - 1$

(2) $\dfrac{x}{\sqrt{1+x} + 1}$

(3) ——

(4) None.

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of

$\Rightarrow$ As at very very small x
By Taylors expansion.

$$\sqrt{1+x}-1$$
$$=(1+x)^{1/2}-1$$
$$=1+\frac{1}{2}x+\frac{(1/2)(1/2-1)}{2!}x^2+\frac{(1/2)(1/2-1)(1/2-2)}{3!}x^3+\dots-1$$
$$=\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3\dots$$

<u>Approximation of functions:-</u>
whenever we are dealing with any non-polynomial functions then their values are approximated upto certain powers of x; i.e. they are approximated by polynomial functions by using taylor's th. & McLaurin's th. upto desired accuracy.

<u>Eg</u> If e^{-x} is approximated by poly. of degree 4 then its value at $1/3$ is ____.

<u>sol:-</u> $e^{-x}=1-x+\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\dots$
$\quad\quad\quad\downarrow\quad\downarrow$
$\quad\quad\quad 1\quad 1/3$

$1/3=0.3333$

$\frac{1}{\lfloor 2}\times\left(\frac{1}{3}\right)^2=\frac{0.3333}{6}=0.0555$

$\frac{1}{\lfloor 3}\left(\frac{1}{3}\right)^3=\frac{0.0555}{9}=0.0061$

$\frac{1}{\lfloor 4}\left(\frac{1}{3}\right)^4=\frac{0.0061}{12}=0.0005$

$=(1+0.0555+0.0005)$
$\quad-(0.3333+0.0061)$
$=\begin{array}{r}1.0560\\-\ .3394\\\hline 0.7166\end{array}$

Numarical Solution of algebraic and Transcendental equation.

In this topic we are interested in finding out approximate solution of $f(x)=0$.
for this purpose several methods are used and in each method approximation ~~and~~ at n^{th} iteration will give some error
If ξ is the actual solution and x_n is it's approximate solution then error in n^{th} approximation is denoted by ϵ_n

$$\epsilon_n = \xi - x_n = \text{Error}$$

$$\epsilon_{n+1} = \xi - x_{n+1}$$

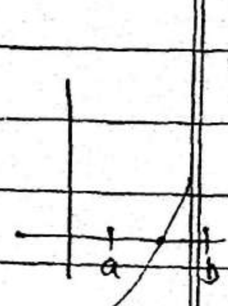
If $\epsilon_{n+1} = \epsilon_n^k (d + \cdots)$ i.e. $\epsilon_{n+1} \approx A\epsilon_n^k$
then order of error in approximation is k

i.e. $\epsilon_{n+1} = 10\epsilon_n^2 + 7\epsilon_n^3 + \cdots$
$$\approx 10\,\epsilon_n^2$$

The simplest and starting method to find approximate solution is Bi-section method.

Bisection method: (Not in CSIR syllabus)

Pre-requisite: If $f(x)$ is continuous function in $[a,b]$ and if $f(a)$ and $f(b)$ are of opposite sine then $\exists$ a root of $f(x)$ between a and b.



The first approximate root by bisection method of $f(x)=0$ in $[a,b]$ is given by $x_1 = \dfrac{a+b}{2}$.

Then the sign of $f(x)$ at x_1 is tasted.
Now we apply the same procedure in interval $[a, x_1]$ or $[x_1, b]$
We repeat this process to get successive approximate value till it is desired.

Comment.

$(n+1)^{th}$ approximate value will lie in the range, $x_{n+1} = \frac{1}{2^n}(b-a)$

So if number of iteration approaches to infinity then length of interval in which approximate sol^n will lie equals to '0' (approaches to 0)

$$x_{n+1} = \lim_{n \to \infty} \frac{1}{2^n}(b-a)$$

$$= 0$$

This method insures convergence of approximate value to the root.

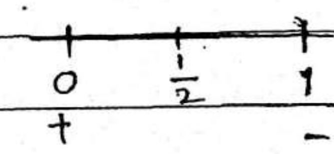
Q. Find the 3rd approximate value of equation $x^4 - 5x + 1 = 0$ lying bet^n 0 & 1. Using Bisection method.

$\Rightarrow$ 1) $f(x)$ is conti.

2) $f(0) = 1$, $f(1) = -3$

3) Both opposite sign so there lies a root bet^n 0 & 1.

$\therefore x_1 = \frac{0+1}{2} = \frac{1}{2}$



$$f\left(\tfrac{1}{2}\right) = \frac{1}{16} - \frac{5}{2} + 1 < 0$$

$$\begin{array}{ccc} 0 & \tfrac{1}{2} & 1 \\ + & - & - \end{array}$$

$$\therefore x_2 = \frac{0 + 1/2}{2} = \frac{1}{4}$$

$$\begin{array}{cccc} 0 & \tfrac{1}{4} & \tfrac{1}{2} & 1 \\ + & - & - & \end{array}$$

$$f\left(\tfrac{1}{4}\right) = \frac{1}{256} - \frac{5}{4} + 1 < 0$$

$$\begin{array}{cccc} 0 & \tfrac{1}{4} & \tfrac{1}{2} & 1 \\ + & - & - & - \end{array}$$

$$x_3 = \frac{0 + 1/4}{2} = \frac{1}{8}$$

Iteration Method. (In CSIR Syllabus)

If we want to find solution of $f(x) = 0$ then we re-write the given equation as ~~$x = \phi(x)$~~. $x = \phi(x)$

Hence solution of $f(x) = 0$ is same as finding fixed point of $\phi(x)$.

In this procedure initial approximate solution $x_0 = 0$.

and the 1st approximate solution is obtained by putting initial approximate

$x_1 = \phi(x_0)$

$x_2 = \phi(x_1)$

Hence by this procedure $(n+1)^{th}$ approximate solution is

$x_{n+1} = \phi(x_n)$. $n = 0, 1, 2, \ldots$

If $f(x) = 0$ is given then there is alot way's to ~~find~~ write $x = \phi(x)$

e.g. $x^2 - x - 1 = 0$

$x = x^2 - 1$

or $x = (x+1)^{1/2}$

or $x - (x^2 - x - 1) = x$

or $x - \dfrac{x^2 - x - 1}{k} = x$ ... k any constant $k \neq 0$

$\vdots$

for $x = x^2 - 1$, $x_0 = 2$

$x_1 = 2^2 - 1 = 3$

$x_2 = 3^2 - 1 = 8$

$\vdots$

diverges to ∞.

Theorem.

If $f(x) = 0$ is written as $x = \phi(x)$ where $\phi(x)$ and $\phi'(x)$ both are continuous in an interval I containing initial approximate value x_0 and if $|\phi'(x)| < 1$ in that interval I then $x = \phi(x)$ will converge to a root of $f(x) = 0$.

Let ξ_1 be the root of $f(x) = 0$ then

$f(\xi_1) = 0$

As $f(x) = 0$ is also $x = \phi(x)$

$\therefore \xi_1 = \phi(\xi_1)$

$x_{n+1} = \phi(x_n)$

$x_1 = \phi(x_0)$

$x_2 = \phi(x_1)$

$\xi_1 - x_1 = \phi(\xi_1) - \phi(x_0)$

$\xi_1 - x_2 = \phi(\xi_1) - \phi(x_1)$

$\vdots$

$\xi_1 - x_{n+1} = \phi(\xi_1) - \phi(x_n)$

ϕ is continuous, derivative exist

$\therefore$ by Lagranges thm

$\xi_1 - x_1 = \phi(\xi_1) - \phi(x_0) = (\xi_1 - x_0)\phi'(\xi_1)$

$\xi_1 - x_2 = \phi(\xi_1) - \phi(x_1) = (\xi_1 - x_1)\phi'(\xi_2)$

$\vdots$

$(\xi_1 - x_n)\phi'(\xi_{n+1})$

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Multiplying left with left & right with right

$(\xi_1 - x_{n+1}) = \cancel{(\xi_1 - x_0)} = (\xi_1 - x_0)\,\phi'(\xi_1) \cdots \phi'(\xi_{1n+1})$

If $\phi'(x) \le K < 1$

$$|\xi_1 - x_{n+1}| = |(\xi_1 - x_0)\,\phi'(\xi_1) \cdots \phi'(\xi_{1n+1})|$$
$$\le K^{n+1}\,|\xi_1 - x_0|$$

As limit $n \to \infty$, RHS $\to 0$ as $K<1$.

$\therefore \lim_{n\to\infty} |\xi_1 - x_{n+1}| = 0$

$\therefore x_{n+1} = \xi_1$

Note

$$\xi_1 - x_{n+1} = \phi'(\xi_{1n+1})(\xi_1 - x_n)$$
$$\epsilon_{n+1} = K\,\epsilon_n$$

error of $(n+1)^{th}$ stage $= K \times$ error at n^{th} stage

~~So order of approximation~~

order of convergence of iteration method is '1'.

Iterative method has linear convergence i.e. 1st order convergence

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Q. If we want to find approximate solution of equation $x^3 - 3x + 1 = 0$. around $1/3$ then which of the following iterative formula will converge to the root ?

1) $x_{n+1} = \frac{x_n^3 + 1}{3}$

2) $x_{n+1} = x_n^3 - 2x_n + 1$

3) $x_{n+1} = (3x_n + 1)^{1/3}$

4) $x_{n+1} = \frac{1}{3 - x_n^2}$

$\Rightarrow$ $\phi(x) = x^3 - 2x + 1$

$\phi'(x) = 3x^2 - 2$

$\phi'(x)|_{1/3} = \left| 3 \frac{1}{9} - 2 \right| = \left| \frac{1}{3} - 2 \right| \not< 1.$

$\therefore$ 2nd is not possible

$\phi(x) = \frac{x_n^3 + 1}{3}$

$|\phi'(x)| = |x_n^2|_{1/3} = \frac{1}{9} < 1.$

$\therefore$ 1st is possible.

$\phi(x) = (3x_n - 1)^{1/3}$

$\frac{1}{3} - 1$ $\quad \phi'(x) = \frac{1}{3} (3x_n - 1)^{-2/3}$

$|\phi'(x)|_{1/3} = \frac{1}{3} \left(\frac{1}{(3x_n - 1)^{2/3}} \right)_{1/3} > 1.$

$\therefore$ not possible

olution
d 1/3
ive
?

$$\phi(x) = \frac{1}{3 - x_n^2}$$

$$\phi'(x) = \frac{-2x_n}{(3 - x_n)^2}$$

$$|\phi'(x)|_{1/3} = \left|\frac{-2/3}{(3 - 1/3)^2}\right| = \frac{2}{64} < 1.$$

$\therefore$ 4th is possible.

If iterative formula in 1st option is used to find approximate value of the root then what is the 1st approximate and 2nd approximate value of the root ?

$$x_0 = \frac{1}{3}$$

$$x_1 = \phi(x_0) = \frac{(1/3)^3 + 1}{3} = \frac{28}{27 \times 3}$$

$$x_2 = \phi(x_1) = \frac{(28/81)^3 + 1}{3}$$

$$x^2 - x - 2 = 0$$

$$x_{n+1} = \sqrt{x_n + 2}$$

$$\phi(x) = \sqrt{x_n + 2}$$

$$|\phi'(x)|_1 = \left|\frac{1}{2\sqrt{x_n + 2}}\right|_1 < 1.$$

Around $x = 1$.

$x_0 = 1$ (+ve root). PAGE NO. DATE / /

This iterative formula will converge to solution of $x^2 - 2x - 2 = 0$ equals to

1) $\sqrt{2}$
2) -1
3) 2
4) $\sqrt{2 + \sqrt{2 + \sqrt{2+2+\cdots}}}$

$\Rightarrow$ Iterative formula $x_{n+1} = \sqrt{x_n + 2}$

$\lim_{n \to \infty} x_n = l$ Suppose

$\therefore \lim_{n \to \infty} x_{n+1} = l$

$\therefore l^2 = l + 2$

$\therefore l^2 - l - 2 = 0$

$(l-2)(l+1) = 0$

$l = 2, -1$

for 4th if $x = \sqrt{2+x}$

$x^2 = 2 + x$

$\Rightarrow x = 2, -1$.

Ans: (3), (4).

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→ like bisection, etc.

Newton - Raphson Method : (In CSIR syllabus)

This method is used to improve the approximate solution of $f(x)=0$ obtained by any one of the method available in the universe. →

If ξ is actual solution of $f(x)=0$ then $f(\xi_1)=0$.

If x_n is approximate solution and x_n+h is actual solution then $f(x_n+h)=0$

$\xi_1 = x_n + h$

$f(\xi_1) = f(x_n+h) = 0$

then by taylors series expansion

$$f(x_n+h) = f(x_n) + hf'(x_n) + \frac{h^2}{2!}f''(x_n) + \cdots = 0$$

then

$$f(x_n+h) \approx f(x_n) + hf'(x_n)$$

By ignoring power of h from 2-onwards we get as h is very small.

$$f(x_n) + hf'(x_n) \approx 0$$

$$\therefore h \approx -\frac{f(x_n)}{f'(x_n)}$$

$$\xi_1 = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

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If we want to find sol^n of $f(x)=0$
Then N-R iterative formula is

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$n = 0, 1, 2, \cdots$
x_0 given

If ξ_1 is the actual root of $f(x) = 0$
then $f(\xi_1) = 0$

$$\Rightarrow f(x_n + (\xi_1 - x_n)) = 0$$

$$f(x_n) + (\xi_1 - x_n) f'(x_n) + \frac{(\xi_1 - x_n)^2}{2!} f''(x_n) + \cdots = 0$$

$$\frac{-f(x_n)}{f'(x_n)} = (\xi_1 - x_n) + \frac{(\xi_1 - x_n)^2}{2!} \frac{f''(x_n)}{f'(x_n)} + \cdots$$

$$x_n - \frac{f(x_n)}{f'(x_n)} = \xi_1 + \frac{(\xi_1 - x_n)^2}{2!} \frac{f''(x_n)}{f'(x_n)} + \cdots$$

$$x_{n+1} - \xi_1 = \frac{(\xi_1 - x_n)^2}{2!} \frac{f''(x_n)}{f'(x_n)} + \cdots$$

$$\epsilon_{n+1} = -\frac{f''(x_n)}{2f'(x_n)} \epsilon_n^2 (1 + \cdots)$$

So order of convergence of Newton Raphson method is '2'

~~Quadratic convergence or second order convergence.~~

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$f(x) = 0$

s

$f(x) = 0$

$f''(x_n) + \cdots = 0$

$\frac{f''(x_n)}{f'(x_n)} + \cdots$

$\frac{''(x_n)}{'(x_n)} + \text{—}$

$+ \cdots$

$\cdots)$

Q If we want the iterative formula to approximate positive square root or square root of a number N then the formula will be ——?

$\Rightarrow$ ~~$f(x)=0$~~ to find sol^n of

let $x = N^{1/2}$

$$x^2 = N$$

$$\therefore x^2 - N = 0$$

$$f(x) = 0$$

$$f(x) = x^2 - N$$

$$f'(x) = 2x$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Q. Consider ~~a~~ iterative formula ~~x_{n+1}~~ $= \frac{2x_n^3 + N}{3x_n^2}$
the given formula is

newton (1) Newton Raphson iterative formula to find cube root of N with 1st order convergence

second (2) " — - - with 2nd order convergence

(3) to find sq. root of N with 2nd order convergence

$x_{n+1} = \frac{2}{3}x_n + \frac{N}{3x_n^2}$

$\Rightarrow x_{n+1} - x_n = \frac{N - x_n^3}{3x_n^2} = -\left[\frac{x_n^3 - N}{3x_n^2}\right]$

$\Rightarrow f(x) = x_n^3 - N$ Now $f(x) = 0 \Rightarrow \boxed{x_n = N^{1/3}}$

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Q. If we want to find first and second approximate value of positive $\sqrt{2}$ by Newton-Raphson method then initial approx value is 1.5 then x_1 and x_2 equals to—

$\Rightarrow x = \sqrt{2}$

$x^2 = 2$

By above

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

$$x_1 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3} \times 2\right)$$

$$= \frac{17}{12}$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right)$$

$$= \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right)$$

$$= \frac{1}{2} \times \frac{17^2 - 12 \times 24}{12 \times 17}$$

$=$

Q. If we want to develope newton Raphson formula for approximating sum of, cube root & square root of any no N then $f(x) = ?$

sol:- $x = N^{1/3} + N^{1/2} \Rightarrow (x - N^{1/2}) = N^{1/3}$

$\{\because 3 > 2\}$ Now $(x - N^{1/2})^3 = N$

$\Rightarrow N = x^3 - 3x^2 N^{1/2} + 3xN - N^{3/2}$

$\Rightarrow N - x^3 - 3xN = -N^{1/2}(3x^2 + N)$

$\Rightarrow (N - x^3 - 3xN)^2 = N(3x^2 + N)$

$\Rightarrow f(x) = 0$

Q.

approx
s to —

Q. find first and second approximate value of $\sqrt[3]{7}$ by taking initial approximate value as 2.

$\Rightarrow$ $x = \sqrt[3]{7}$

$x^3 - 7 = 0$

$f(x) = x^3 - 7$

$f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 7}{3x_n^2}$$

$$= \frac{2x_n^3 + 7}{3x_n^2}$$

$$= \frac{2}{3}x_n + \frac{7}{3x_n^2}$$

$$= \frac{1}{3}\left[2x_n + \frac{7}{x_n^2}\right]$$

$x_0 = 2$

$$x_1 = \frac{1}{3}\left[4 + \frac{7}{4}\right] =$$

$$x_1 = \frac{23}{3}$$

$$x_2 = \frac{1}{3}\left[2 \times \frac{23}{3} + 7 \times \frac{9}{(23)^2}\right]$$

Solution of system of linear equations:

Here ~~I~~ we have n-equations in n-unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

When coefficient matrix is nonsingular then system will have unique solution. In Numarical analysis we have two methods of finding solution.

1) Direct method (Exact sol^n)
2) Indirect / Iterative method (Approx sol^n)

Direct method

A) Matrix Inversion method ($AX = B \Rightarrow X = A^{-1}B$)
B) Gauss elimination method
C) Gauss diversion method (Not in syllabus)

B) Gauss Elimination method:

In this method we form augumented matrix $(A|B)$.
In the 1st step by using elementory row transformation we make all the elements in the 1st column below the 1st row equals to zero
In the 2nd step we make all the elements in 2nd column below 2nd row equal to zero using elementary row transformation.

proceeding in similar fashion in the i^{th} step ~~all~~ we make all the elts below the i^{th} row equals to zero we go on doing this upto $(n-1)^{th}$ step Thus principle part reduces to upper triangular matrix and from bottom to top we evaluate the value of one variable at one step and thus solution is acheived.

Note

If at the i^{th} step all the elements in i^{th} column except in the i^{th} row is made zero then the method will be known as guass jeordon method but this has to go upto n^{th} step and principle part ~~part~~ becomes diagonal matrix, and parallely the value of each variable will be obtained in one step.

Q. Apply Gauss elimination method

$$x + y - z = 1$$
$$x - y + z = 1$$
$$-x + y + z = 1$$

to find solution of the following system of equation.

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$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right] \Rightarrow \begin{array}{l} z = 1 \\ -y+z=0 \Rightarrow \underline{y=1} \\ x+y-z=1 \Rightarrow \underline{x=1} \end{array}$$

Indirect / Iterative method.

If we have system of equation $Ax = B$ then we take A in such a way that all the diagonal entries of A are nonzero then from the 1st equation we write the value of x_1 in terms of remaining variables.

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}x_2}{a_{11}} - \frac{a_{13}}{a_{11}}x_3 - \cdots - \frac{a_{1n}}{a_{11}}x_n$$

from 2nd eqn we write

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}x_3}{a_{22}} - \cdots - \frac{a_{2n}}{a_{22}}x_n$$

proceeding in similar fashion the nth eqn writes x_n (nth variable

$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}x_2}{a_{nn}} - \cdots - \frac{a_{n\,n-1}}{a_{nn}}x_{n-1}$$

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In <u>Jacobi's method</u> (Not in syllabus)
$(k+1)^{th}$ approximate value of each variable is obtained by putting k^{th} approximate value in the above system of eqn on RHS

$$x_1^{k+1} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^k - \dots - \frac{a_{1n}}{a_{11}} x_n^k$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^k - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^k$$

In <u>Gauss-saidel method</u> (In syllabus)
at each step of evaluation of the value of variables the latest evaluated value of variable on RHS are taken

$$x_1^{k+1} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^k - \dots - \frac{a_{1n}}{a_{11}} x_n^k$$

$$x_2^{k+1} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{k+1} - \dots - \frac{a_{2n}}{a_{nn}} x_n^k$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{k+1} - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^{k+1}$$

Note

If 0^{th} or initial iterative value of variables are not given then they are taken as '0'.

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Note.

The above methods converges to the actual solution if $\sum_{\substack{j=1 \\ i \neq j}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \leq 1 \ \forall i$

and in atleast one case $\sum_{1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| < 1$

Q. Consider the system of equations

$$10x + y - 2z = 9$$
$$x + 10y + z = 12$$
$$x - 3y + 10z = 8$$

for this system of equation write down iterative formula to evaluate approximate value of x, y, z.

$\Rightarrow$ $x = \frac{9}{10} - \frac{1}{10} y + \frac{2}{10} z$

$x = 0.9 - 0.1 y + 0.2 z$

$y = 1.2 - 0.1 x - 0.1 z$

$z = 0.8 - 0.1 x + 0.3 y$.

$x^{n+1} = 0.9 - 0.1 y^{n} + \frac{2}{10} z^{n}$

$y^{n+1} = 1.2 - 0.1 x^{n+1} - 0.1 z^{n}$

$z^{n+1} = 0.8 - 0.1 x^{n+1} + 0.3 y^{n+1}$

Q. find 1^{st} and 2^{nd} iterative value of x, y, z by Gauss Seidel method for the above eqns.

1^{st} iterative value -

$x^{1} = 0.9$

$y^{1} = 1.2 - (0.1)(0.9)$

$= 1.11$

$z^{1} = (0.8) - (0.1)(0.9) + (0.3)(1.11)$

$= 1.043$

2^{nd} iterative value of

$x = (0.9) - (0.1)(1.11) + (0.2)(1.04)$

$= 0.9976$

$y = (1.2) - (0.1)(0.9976) + (0.1)(1.043)$

$= 0.99594$

$z = (0.8) - (0.1)(0.9976) + (0.3)($

≈ 1

x	x	y	z
1^{st}	0.9	1.11	1.043
2^{nd}	0.9976	0.99594	≈ 1

Q Consider the system of eqns given by

$$\begin{pmatrix} 0.99 & 1.01 \\ 1.01 & 0.99 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

If approximate value of solution is taken as $\hat{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ then find error and residue in computation.

⇒ True value $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\text{Error } (E) = x - \hat{x}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Residue : $A(x - \hat{x}) = \begin{pmatrix} 0.99 & 1.01 \\ 1.01 & 0.99 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 0.02 \\ -0.02 \end{pmatrix}$$

$$= B - \hat{B}$$

$$= \begin{pmatrix} 1\% \\ -1\% \end{pmatrix}$$

0.02 out of 2

-0.02 out of 2.

Comments

In the computation of X, Error is very very large but residue is very small.

Note.

There is no relationship betⁿ error and residue in approximation of solution of system of linear equation.

That means small error may result into larger or smallar residue or larger error may result into smaller or larger residue that all depends upon the coefficient

Interpolation

If we are dealing with function of single variable then if we are given certain set of ordered pair of dependent or independent variable i.e. (x_i, y_i) also known as set of tabulated points.

If we have set of $(n+1)$ - tabulated points (x_i, y_i), $i = 0, 1, \dots n$, such that $x_0 < x_1 < x_2 < \dots < x_n$ then in this chapter we are interested in to fit a polynomial curve $y = f(x)$ which satisfy $y_i = f(x_i)$ which is further used to find approximate value of y for any x lying in the interval $[x_0, x_n]$. This process is known as interpola^n. and the fitted curve is known as interpolating curve. If this curve is used to find the value of y for x outside the interval then the process is known as extrapolation

$$x \in (-\infty, x_0) \cup (x_n, \infty)$$

$$y = a_0 + a_1 x + \dots + a_n x^n$$

$$y_0 = a_0 + a_1 x_0 + \dots + a_n x^n$$

$$\vdots$$

$$y_n = a_0 + a_1 x_n + \dots + a_n x_n^n$$

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & & x_n^n \end{bmatrix}$$ Van-der mode matrix.

If we have $(n+1)$-tabulated points then we can fit a polynomial of degree upto-n.
n-tabulated points $\leq$ poly. of degree $(n-1)$
5 - tabulated points $\leq$ 4 degree poly.
By direct method of finding interpolating curve assume $y = a_0 + a_1 x + \cdots + a_n x^n$ and on the RHS we put $x = x_i$ and on LHS we put $y = y_i$ to get $n+1$ linear equations in a_i's which are also $n+1$ in numbers whose coefficient matrix is nonsingular (Because determinant of coefficient matrix is Van-der mode).
Hence unique value of a_i's will be obtained
Thus interpolating curve will be obtained.

Another Direct method.
Another direct method to find y is to write y as,

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1}).$$

At $x = x_0$, $y = a_0$
$x = x_1$, $y = a_1$
$\vdots$
$x = x_i$, $y = a_i$
$\vdots$
$x = x_n$, $y = a_n$

Q. find the interpolating curve corresponding to the following table

x	0	1	2
y	1	0	-1

$\Rightarrow$ Method: (1)

points are $(0,1)$, $(1,0)$, $(2,-1)$

We can fit the curve of degree ≤ 2

Assume

$$y = a + bx + cx^2$$

$(0,1) \Rightarrow 1 = a$

$(1,0) \Rightarrow 0 = a + b + c$

$(2,-1) \Rightarrow -1 = a + 2b + 4c$

for $a = 1$, $\left.\begin{array}{l} b + c = -1 \\ 2b + 4c = -2 \end{array}\right\} \Rightarrow c = 0 \Rightarrow b = -1$

$\therefore a = 1,\ b = -1,\ c = 0$

$\therefore y = a + bx + c$

$\therefore \boxed{y = 1 - x}$ is the required interpolating curve.

Method (2)

$$y = a + b(x-0) + c(x-0)(x-1)$$

At $x = 0$, $1 = a$

At $x = 1$, $0 = 1 + b \Rightarrow b = -1$

At $x = 2$, $c = 0$

$$\therefore \boxed{y = 1 - x}$$

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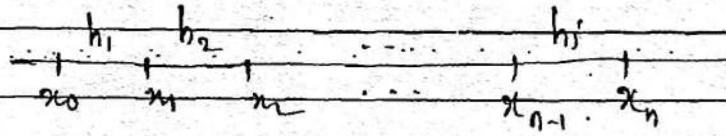
Finite Difference.

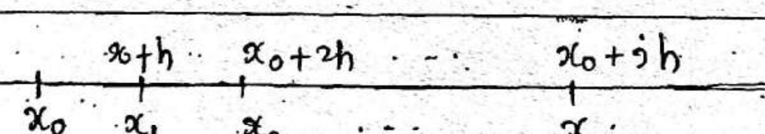
If we have set of $(n+1)$ - tabulated points where each successive points at finite difference then the distance between two successive points will be either constants or variable. ~~or either~~ in erlier case the set of points are known as evenly spaced and in the later case it is known as unevenly spaced points

if $h_i = x_i - x_{i-1}$



Evenly spaced set of points:

If we have set of tabulated points (x_i, y_i) then if $x_k - x_{k-1} = h \quad \forall \ K = 1, 2 \ldots$ then h is known as Steplength of tabulated points



$$\therefore \ x_i = x_0 + ih \ , \quad i = 0, 1, \ldots, n$$

In these cases Newton sahab has introduced three operators to deal with the functional values of these tabulated points

1) Δ : Forward difference operator. (In Syllabus)
2) ∇ : Backward difference operator (In Syllabus)
3) δ : Central difference operator. (Not in Syllabus)

Forward difference operator of y_i

$$\Delta y_i = y_{i+1} - y_i$$

Backword difference operator of $y_i = y_i - y_{i-1}$

$$\nabla y_i = y_i - y_{i-1}$$

Central difference operator of y_i

$$\delta y_{i+\frac{1}{2}} = y_{i+1} - y_i$$

$$\begin{aligned}
\Delta^2 y_i &= \Delta(\Delta y_i) \\
&= \Delta(y_{i+1} - y_i) \\
&= y_{i+2} - y_{i+1} - (y_{i+1} - y_i) \\
&= y_{i+2} - 2y_{i+1} + y_i
\end{aligned}$$

$$\begin{aligned}
\nabla y_{i+1} &= y_{i+1} - y_i \\
&= \Delta y_i
\end{aligned}$$

$$\begin{aligned}
\Delta^3 y_i &= \Delta^2(\Delta y_i) \\
&= \Delta^2(y_{i+1} - y_i) \\
&= \Delta(\Delta y_{i+1} - \Delta y_i) \\
&= \Delta(y_{i+2} - y_{i+1} - y_{i+1} + y_i) \\
&= y_{i+3} - y_{i+2} - 2(y_{i+2} - y_{i+1}) + y_{i+1} - y_i \\
&= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i \\
&= y_{i+3} - 3(y_{i+2} - y_{i+1}) - y_i
\end{aligned}$$

Shift operator (E)

$$E y_i = y_{i+1}$$

$$E^k y_i = y_{i+k}$$

Relation between E and Δ is that

$$\Delta = E - 1$$

$$\Delta^k = (E-1)^k$$

$$= E^k - {}^kC_1 E^{k-1} + \cdots + {}^kC_2 E^{k-2} + \cdots + 1$$

$$\Delta y_i = y_{i+1} - y_i = (E-1) y_i$$

Q. By using this concept obtain $\Delta^3 y_i$

$\Rightarrow$

$$\Delta^3 y_i = (E-1)^3 y_i$$

$$= (E^3 - 3E^2 + 3E - 1) y_i$$

$$= E^3 y_i - 3E^2 y_i + 3E y_i - y_i$$

$$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

If we have $(n+1)$ - tabulated points:

x	y	Δ	Δ^2	Δ^3	...
x_0	y_0	y_0			
x_1	y_1	y_1	y_0	y_0	
x_2	y_2	y_2	y_1		
x_3	y_3				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	y_{n-2}	
x_{n-1}	y_{n-1}	y_{n-1}	y_{n-2}		
x_n	y_n				

Same table used for inverted ∇ operator:

x	y	∇	∇^2	∇^3
x_0	y_0			
x_1	y_1	y_1	y_2	y_3
x_2	y_2	y_2	y_3	$\vdots$
x_3	y_3	y_3		y_n
$\vdots$	$\vdots$	$\vdots$	y_n	
x_n	y_n	y_n		

Q.

x	0	1	2	3
y	-1	2	-1	3

Convert given table into operator difference table.

$\Rightarrow$

x	y	Δ/∇	Δ^2/∇^2	Δ^3/∇^3
0	-1			
		3		
1	2		-6	
		-3		13
2	-1		7	
		4		
3	3			

Newtons Interpolation formulas:

For set of tabulated points (x_i, y_i), $i = 0, 1, \dots n$ which are evenly space,

$$x_k - x_{k-1} = h \quad : \text{Steplength}$$

We write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots$$
$$\dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

At $x = x_1$

$$\frac{y_1 - y_0}{x_1 - x_0} = a_1$$

$$\frac{\Delta y_0}{h} = a_1$$

At $x = x_2$

$$y_2(x_2) = y_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$= y_0 + \frac{\Delta y_0}{h}(x_2 - x_0) + a_2(2h)(h)$$
$$= y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2!)h^2$$
$$= y_0 + 2\Delta y_0 + a_2(2!)h^2$$

$$\therefore \frac{y_2 - 2(y_1 - y_0) - y_0}{h^2(2!)} = a_2$$

$$\therefore \frac{\Delta^2 y_0}{2!\, h^2} = a_2$$

$$\vdots$$

$$\frac{\Delta^n y_0}{(n!)\, h^n} = a_n$$

$$\therefore y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{h^2(2!)}(x - x_0)(x - x_1)$$

$$+ \cdots + \frac{\Delta^n y_0}{h^n(n!)}(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Let $x = x_0 + ph$

$$x - x_0 = ph$$

$$x - x_0 - x_1 + x_1 = ph$$

$$x - x_1 + x_1 - x_0 = ph$$

$$x - x_1 + h = ph$$

$$x - x_1 = ph - h$$

$$x - x_1 = (p-1)h$$

Similarly $x - x_2 = (p-2)h$

$$\vdots$$

$$x - x_i = (p-i)h$$

$$\therefore y(x) = y_0 + \frac{\Delta y_0}{h} ph + \frac{\Delta^2 y_0}{h^2(2!)} ph \times (p-1)h + \cdots$$

$$\cdots + \frac{\Delta^n y_0}{h^n(n!)} ph \times (p-1)h \times (p-2)h \times \cdots (p-\overline{n-1})h$$

$$\therefore y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots$$

$$\cdots + \frac{p(p-1)\cdots(p-\overline{n-1})}{n!}\Delta^n y_0 .$$

This is known as newtons forward difference interpolation formular.

$x = x_0 + h$, h : steplength

x_0 is initial or starting value.

Now

Let $y = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \cdots$

$\cdots + a_n(x-x_n)(x-x_{n-1})\cdots(x-x_1)$

At $x = x_n$; $y_n = a_0$

$x = x_{n-1}$; $y_{n-1} = y_n + a_1(-h)$

$$a_1 = \frac{\nabla y_n}{h}$$

$$a_2 = \frac{\nabla^2 y_n}{h^2(2!)}$$

$$\therefore y = y_n + \frac{\nabla y_n}{h}(x-x_n) + \frac{\nabla^2 y_n}{h^2(2!)}(x-x_n)(x-x_{n-1})$$

$$+ \cdots + \frac{\nabla^n y_n}{h^n(n!)}(x-x_n)(x-x_{n-1})\cdots(x-x_1).$$

$x - x_n = ph$

$x - x_{n-1} = (p+1)h$

$\vdots$

$x - x_{n-i} = (p+i)h$

$$\therefore y = p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \cdots + \frac{p(p+1)(p+2)\cdots(p+\overline{(n-1)})}{n!}\nabla^n y_n .$$

This is known as Newtons Backward interpol^n

Q. Consider set of points

x	0	1	2
y	-1	0	1

By newtons forward and Backward formula find interpolating curve

$\Rightarrow$

x	y	Δ / ∇	Δ^2 / ∇^2
0	-1		
		1	
1	0		0
		1	
2	1		

Here $h = 1$, $x_0 = 0$, $x_n = 2$

By Newtons forward interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \dots + \frac{p(p-1)\dots p-(n-1)}{n!}\Delta^n y_0$$

$$y = -1 + p(1) + \frac{p(p-1)}{2!} \times 0$$

$$y = p - 1 \qquad \text{As } x = x_0 + ph$$

$$p = x - x_0$$

$$\therefore \boxed{y = x - 1} \qquad p = x$$

By Newtons Backward interpolation formula,

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \dots$$

$$y = +1 + p(1) + 0$$

$$y = p + 1$$

As $x - x_n = ph$

$$x - x_n = p$$

$$x - 2 = p$$

$$\therefore y = x - 2 + 1$$

$$\therefore \boxed{y = x - 1}$$

Comment.

If difference operators have upto k^{th} power equal to nonzero and rest highest powers are zero then it will interpolate curve of degree k.

find

Q. The set of data (1, 2); (2, 5), (3, 10), (4, 17), (5, 26), (6, 37) will interpolate curve of degree upto ____ ?

$\Rightarrow$

x	y	Δ	Δ^2	Δ^3
1	2	3	2	0
2	5	5	2	0
3	10	7	2	0
4	17	9	2	
5	26	11		
6	37			

$2(n-1)\Delta^n y_0$

$$\Delta^2 \neq 0 \quad , \Delta^3 = \Delta^4 = 0.$$

$\therefore$ By above comment, the given sets of data will interpolate a curve of degree upto $\underline{2}$.

u,

$$y = 2 + p(3) + \frac{p(p-1)}{2}(2)$$
$$= 2 + 3p + p^2 - p$$
$$= 2 + 2p + p^2$$
$$= 2 + 2(x-1) + (x-1)^2$$
$$= 2 + 2x - 2 + x^2 - 2x + 1$$
$$\boxed{y = x^2 + 1}$$

$x = x_0 + ph$

$x = 1 + p$

$p = x - 1$

Q.

Lagranges Interpolation Formulas :

If we have unevenly set of points (x_j, y_j), $j = 0, 1, \ldots, n$

$$y(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n + \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots$$

$$\vdots$$

$$L_n(x) = \sum_{j=0}^{n} \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_{n-j})} y_j$$

Q.

$$= \sum_{j=1}^{n} l_j(x)\, y_j$$

This method is valid for set of tabulated points which is either evenly spaced or unevenly spaced.

$$\begin{array}{cccccc}
1 & a_0 & a_1 & a_2 & \cdots & a_n \\
y & a_0 & a_1 x & a_2 x^2 & \cdots & a_n x^n \\
y_0 & a_0 & a_1 x_0 & a_2 x_0^2 & \cdots & a_n x_0^n \\
\vdots & & & & & \\
y_n & a_0 & a_1 x_n & a_2 x_n^2 & \cdots & a_n x^n
\end{array}$$

eliminating $1, a_0, a_1, \ldots, a_n$ from 1^{st}, 2^{nd}, $(n+2)^{th}$ column we get

$$\begin{vmatrix} 1 & 1 & x & \cdots & x^n \\ y & 1 & x_0 & & x_0^n \\ y_0 & 1 & x_1 & & x_1^n \\ y_1 & 1 & x_2 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ y_n & 1 & x_n & & x_0^n \end{vmatrix} = 0$$

Van-dar mode determinant.

Q. Consider set of points

x	0	1	2
y	-1	0	1

By Lagranges method find interpolating curve of the set of database.

$\Rightarrow$
$$y = \frac{(x-1)(x-2)}{(0-1)(0-2)}(-1) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(0) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(1)$$

$$= -\frac{(x-1)(x-2)}{2} + \frac{x(x-1)}{2}$$

$$= \frac{(x-1)}{2}[x - x + 2]$$

$$y = x - 1$$

Q. If we want to find the values at ??

x	1	2	3	4
y	7	11	??	19

Then we can use ...

1) Newton Raphson Interpolation formula.
2) Newton Forward Interpolation formula.
3) Lagranges Interpolation formula.
4) All.

$\Rightarrow$ Given set of points are $(1,7)$, $(2,11)$, $(4,19)$ which are not evenly spaced

Therefore,

Newtons formula invalid here

$\therefore$ Only Lagranges interpolation formula is applicable.

$\therefore$ Ans: (3).

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Q.

x	1	2	$2\frac{1}{2}$	3
y	7	11	??	19

Then we can use,

1) Newton raphson formula.
2) Newton Forward Interpolation formula.
3) Lagranges formula
4) All.

⟹ Here the points are evenly spaced
So newtons formula is applicable.
∴ All the methods are applicable

Ans : (4) For SAQ.
: (1), (2), (3), (4) for MAQ.

Hermite Interpolation formula.

If we have set of tabulated points in which the value of independent variable alongwith corresponding value of the independent variable as well as with it's derivative, that means we should have $(n+1)$ points.

(x_i, y_i, y_i') ; $i = 0, 1, 2, \cdots, n$.

Then our goal is to fit polynomial of least degree which satisfy

$$H_{2n+1}(x_i) = y_i$$

$$H'_{2n+1}(x_i) = y'_i \qquad i = 0, 1, 2, \cdots, n.$$

then we can fit a polynomial of degree upto $(2n+1)$.

Q Consider set of tabulated points $(1,2,2)$ and $(0,1,0)$, find $H_3(x)$ which is

$\Rightarrow$ fitted over dataset

$\Rightarrow$ Let us assume $H_3(x) = a + bx + cx^2 + dx^3$

$1 \quad 2 \quad 2 \qquad\qquad 0 \quad 1 \quad 0$

$$H_3(x) = a + bx + cx^2 + dx^3$$

$$H_3'(x) = b + 2cx + 3dx^2$$

At $(1, 2, 2)$ $x_i \; y_i \; y_i'$	At $(0, 1, 0)$ $x_i \; y_i \; y_i'$
$2 = a + b + c + d \quad \text{—} (1)$	$1 = a \quad \text{—} (2)$

at $x_1 = 1$, $y_1' = 2$

$\therefore 2 = b + 2c + 3d$

and at $x_2 = 0$, $y_2' = 0$

$\therefore 0 = b \quad \text{—} (3)$

$\therefore$ from (1), (2), (3)

$$2 = 1 + 0 + c + d$$

$$\therefore c + d = 1 \quad \text{—} (4)$$

Also $2c + 3d = 2 \quad \text{—} (5)$

Solving (4) and (5) we get

$$-d = -1 \Rightarrow d = 1$$

$$\Rightarrow c = 0$$

$$\therefore H_3(x) = a + bx + cx^2 + dx^3$$

$$= 1 + x^3$$

Remark: For 'n' tabulated points degree of Hermite polynomial will be upto $2n-1$.
For $(n+1)$ - tabulated points degree will be upto $2n+1$.

Spline Interpolation.

In general fitting a curve for a whole interval resultant into much error in approximation at points in the interval. So it was decided to fit local curves in the interval $[x_0, x_n]$ where one curve will be for $[x_0, x_1]$ another curve will be for $[x_1, x_2]$ and i^{th} curve will be for $[x_{i-1}, x_i]$, n^{th} curve will be for $[x_{n-1}, x_n]$ and each of such curve should agree at their end points. The collection of all such curve is interpolating curve and in each subinterval the fitted curve are known as spline curves.

$$S(x) = \begin{cases} S_1(x) & ; \quad x_0 \le x \le x_1 \\ S_2(x) & ; \quad x_1 \le x \le x_2 \\ S_3(x) & ; \quad x_2 \le x \le x_3 \\ \vdots & \\ S_i(x) & ; \quad x_{i-1} \le x \le x_i \\ \vdots & \\ S_n(x) & ; \quad x_{n-1} \le x \le x_n \end{cases}$$

There are lot of spline curves are available simplest one is linear spline curve, another one is quadratic spline curve, cubic spline curve etc...

If we are fitting n^{th} degree spline curves then to fit the curve we use $S(x), S'(x), S''(x), \ldots, S^{n-1}(x)$ as continue in $[x_0, x_n]$.

Q. Consider the set of points (1,2), (2,5), (4,7). By using linear spline find value of y at 3 and also value of y at 1.5.

$\Rightarrow$

Spline function $S(x) = \begin{cases} 3x-1 & ; 1 \le x \le 2 \\ x+3 & ; 2 \le x \le 4 \end{cases}$

$y - y_1 = m(x - x_1)$ $\quad \boxed{m = \dfrac{y_2 - y_1}{x_2 - x_1} = 3}$

$y - 2 = m(x - 1)$

$y - 2 = 3x - 3$ $\quad$ Also $y - 5 = 1(x - 2)$

$y = 3x - 1$ $\quad\quad y = x + 3$

$\because$ $3 \in [2, 4]$

$\therefore$ $y(\text{at } 3) = 3 + 3 = 6$

and $1.5 \in [1, 2]$

$\therefore$ $y(\text{at } 1.5) = 3(1.5) - 1$

$= 3.5$

Q. Consider set of points (1,0), (2,1) & (4,−1). By using quadratic spline interpolation formula find y(3) and y(1.5)

$\Rightarrow$ Let

$S(x) = \begin{cases} a + bx + cx^2 & ; 1 \le x \le 2 \\ d + ex + fx^2 & ; 2 \le x \le 4 \end{cases}$

At $x = 1$, $y = 0$

$\therefore 0 = a + b + c$ ——— (1)

At $x = 2$, $y = 1$

$1 = a + 2b + 4c$ —(2) & $1 = d + 2e + 4f$ —(3)

At $x = 4$, $y = -1$

$-1 = d + 4e + 16f$ —(4)

Also

$$S_1'(x) = \begin{cases} b + 2cx & 1 \le x \le 2 \\ e + 2fx & 2 \le x \le 4 \end{cases}$$

Left hand derivative at $x=2$ must be equal to Right hand derivative at $x=4$ because S and S' should be continuous

$\therefore \quad b + 4c = e + 4f$ — (5)

Also from $S(x)$

RHL = LHL by continuity

$\therefore \quad a + 2b + 4c = d + 2e + 4f$ — (6)

Solving equation (1) to (6) we get

st be
at $x=4$
ntinuous

– Numarical Differentiation –

It we have set of tabulated points (x_i, y_i) $i=0,1,2,\cdots,n$ then by Newtons forword interpolation formula for evenly spaced set of points :

NFDI :

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \cdots \quad x = x_0 + ph.$$

NBDI :

$$y = y_n + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \cdots \quad x = x_n + ph.$$

In Numarical anelysis we are interested in approximate finding out $(dy/dx)|_{x=x_0}$ and $(dy/dx)_{x=x_n}$.

$$\text{As} \quad \frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$

$$\text{as} \quad x = x_0 + ph$$

$$dx = h\,dp \quad \therefore dp/dx = 1/h$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\frac{dy}{dp}$$

$$= \frac{1}{h}\left\{\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \cdots\right.$$

At $x = x_0 \Rightarrow p = 0$

$$\therefore \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0}$$

$$= \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2!}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right.$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0}$$

$$= \frac{1}{h}\left\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right\}$$

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Q. If in a company production of football in Lakhs years observed as follows:

Years	Production
2010	5
2011	10
2012	8
2013	7
2014	9

What is the annual rate of production of football in year 2010 and year 2014 approximated by the given data.

⇒

x	y	Δ/∇	Δ^2/∇^2	Δ^3/∇^3	Δ^4/∇^4
2010	5				
		5			
2011	10		-7		
		-2		8	
2012	8		1		-6
		-1		2	
2013	7		3		
		2			
2014	9				

$$\left(\frac{dy}{dx}\right)_{x=x_0=2010} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right\}$$

$$= \frac{1}{1}\left\{5 - \frac{1}{2}(-7) + \frac{1}{3}(8) - \frac{1}{4}(-6)\right\}$$

$$= \frac{38}{3}$$

$$\left(\frac{dy}{dx}\right)_{x=x_n=2014} = \frac{1}{1}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n\right)$$

$$= 2 + \frac{1}{2}(3) + \frac{1}{3}(2) + \frac{1}{4}(-6)$$

$$= \frac{8}{3}$$

Lakhs

Negative $\Rightarrow$ Negative rate of growth.
+ve value $\Rightarrow$ Positive rate of growth.

cn of

If we want to find approximate value of $f'(x_0)$ then if we have data available at one step ahead of x_0 and one step before x_0 then it may be approximated by

$$\frac{f(x_0+h)-f(x_0-h)}{2h} \approx f'(x_0)$$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} \quad \dots \frac{0}{0} \text{ form}$$

$$\lim_{h\to 0} \frac{f'(x_0+h)+f'(x_0-h)}{h}$$

Also for 2-step ahead, 2 step before

$$f'(x_0) \approx \frac{f(x_0+2h)-f(x_0-2h)}{4h}$$

Q. If we approximate $f'(x)$ by above formula then error in approximation is of order — and error in approximation is approximately —

$$\Rightarrow f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

Now,

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots$$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \cdots$$

$$\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{2hf'(x_0)}{2h} + \frac{2(h^3/3!)f'''(x_0)}{2h} + \cdots$$

$$= f'(x_0) + \frac{h^2}{3!}f'''(x_0) + \cdots$$

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Error in approximations

$$E = f'(x_0) - \left(\frac{f(x_0+h) - f(x_0-h)}{2h} \right)$$

$$= \frac{-h^2}{6} f'''(x_0)$$

$\therefore$ order of error is order of h^2 i.e $= 2$.

Error is $\frac{-h^2}{6} f'''(x_0)$.

$f(x_0) = f_0$.

$f(x_0-h) = f_{-1}$.

$f(x_0+h) = f_1$.

$f(x_0+ih) = f_i$.

i.e $= 2$.

Numarical Integration -

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If we have set of tabulated points (x_i, y_i), $i = 0, 1, \cdots, n$ then we can fit a curve of degree upto n.

$$y(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$+ \cdots + \left(\frac{P(P-1)\cdots(P-\overline{(n-1)})}{n!}\right)\Delta^n y_0$$

On the basis of given set of tabulated pts we are interested in finding out $\int_{x_0}^{x_n} y\,dx$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \cdots + \int_{x_{n-1}}^{x_n} y\,dx .$$

We are interested in finding out approximate value of integral. We divide integral into sum of integrals where each subintegral takes fixed number of tabulated points and then over tabulated points of sub-interval we fit local curve of suitable degree in each of them and we evaluate approximate value of the integral by this process.

1) **Trapezoidal rule of approximation:**

It consists of two tabulated points in each of the subinterval over which integration is approximated.

$$x = x_0 + Ph$$

$$dx = h\,dp$$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \cdots + \int_{x_{n-1}}^{x_n} y\,dx$$

In interval $[x_0, x_1]$

x	y	Δ
x_0	y_0	y_0
x_1	y_1	

$$y(x) = y_0 + P(y_1 - y_0)$$

$$\int_{x_0}^{x_1} y\,dx = h\int_{x_0}^{x_1} (y_0 + Py_1 - Py_0)\,dp$$

$$\simeq h\int_0^1 [y_0 + P(y_1 - y_0)]\,dp$$

$$\simeq h\left[y_0 + \frac{P^2}{2}(y_1 - y_0)\Big|_0^1\right]$$

$$\simeq h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$\simeq \frac{h}{2}[y_0 + y_1]$$

Similary, $\int_{x_1}^{x_2} y\,dx \simeq \frac{h}{2}[y_1 + y_2]$

$\vdots$

$$\int_{x_{n-1}}^{x_n} y\,dx \simeq \frac{h}{2}[y_{n-1} + y_n]$$

Adding all these n-integrals we get,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right\}.$$

OR

$$\int_{x_0}^{x_n} y\,dx = h\left\{\frac{1}{2}(y_0+y_n) + (y_1+y_2+\cdots+y_{n-1})\right\}$$

= distance betn two consicutive ordinates × {mean of 1st and last ordinates + Sum of all intermediate ordinates}

This rule is known as Trapezoidal rule.

Now

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1}\left(y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots\right)dx$$

(y About $x = x_0$)

As

$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots$$

$$\therefore \int_{x_0}^{x_1}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots\right]dx$$

$$= hy_0 + \frac{h^2 y_0'}{2!} + \frac{h^3}{6}y_0'' + \cdots$$

$\therefore$ Actual value of integration:

$$\int_{x_0}^{x_1} y\,dx = hy_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{6}y_0'' + \cdots$$

Approximate value of integration is.

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0+y_1)$$

$$= \frac{h}{2}\left\{y_0 + \left(y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots\right)\right\}$$

$$= \frac{h}{2}\left\{2y_0 + hy_0' + \frac{h^2 y_0''}{2!} + \frac{h^3}{3!}y_0''' + \cdots\right\}$$

$$= hy_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{4}y_0'' + \frac{h^4}{2\cdot 3!}y_0''' + \cdots$$

$\frac{h}{2}(y_{n-1}+y_0)$

$\ldots)\}.$

Therefore

Error = Actual value − Approximate value

$$= \left[\int_{x_0}^{x_1} y\, dx \right]_{Act} - \left[\int_{x_0}^{x_1} y\, dx \right]_{Approx}$$

$$= \cancel{hy_0} + \cancel{\frac{h^2}{2!} y_0'} + \frac{h^3}{6} y_0'' + \frac{h^4}{24} y_0''' + \cdots$$

$$- \left\{ \cancel{hy_0} + \cancel{\frac{h^2}{2!} y_0'} + \frac{h^3}{4} y_0'' + \frac{h^4}{12} y_0''' + \cdots \right\}$$

$$= h^3 \left(\frac{1}{6} - \frac{1}{4} \right) y_0'' + h^4 \left(\frac{1}{24} - \frac{1}{12} \right) y_0''' + \cdots$$

$$= \frac{-h^3}{12} y_0'' - \frac{h^4}{24} y_0''' + \cdots$$

$$\therefore \{\text{Error}\}_{[x_0, x_1]} = -\frac{h^3}{12} y_0''$$

$\therefore$ Error in the whole interval $[x_0, x_n]$ is

$$\{\text{Error}\}_{[x_0, x_n]} = \frac{-h^3}{12} \{ y_0'' + y_1'' + y_2'' + \cdots y_{n-1}'' \}$$

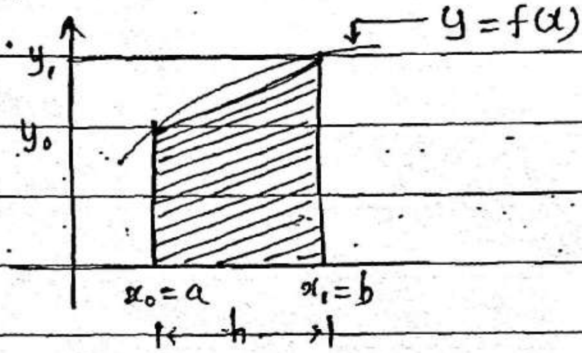
$$< n \times \left(\frac{-h^3}{12} y_{max}'' \right)$$

As $nh = b - a = (x_n - x_0)$

$$\text{Error} \approx \frac{-(b-a)}{12} h^2 y_{max}''$$

This is error in approximation in the trapezoidal rule of order 2

Q. The geometrical significance of this rule is that the curve $y=f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1) and $(x_2, y_2) \cdots (x_{n-1}, y_{n-1})$ and (x_n, y_n). The area bounded by the curve $y=f(x)$, the ordinates $x=x_0$ and $x=x_n$ and the x-axis is then approximately equivalent to the sum of the areas of n Trapeziums obtained.



Shaded Area = Area of Trapezium $\approx \int_a^b f(x)\,dx$

Q. If we want to find the approximate value of $\int_0^1 x^2\,dx$ by using trapezoidal rule and taking steplength of $1/3$ then it's approximate value will be ——?
Also find error in approximation.

$\Rightarrow$

x	0	$1/3$	$2/3$	$3/3=1$
$y=x^2$	0	$1/9$	$4/9$	1

$$\int_0^1 x^2\,dx \approx \frac{1/3}{2}\left\{(0+1)+2\left(\frac{1}{9}+\frac{4}{9}\right)\right\}$$

$$\approx \frac{19}{54}$$

$$\text{Error} = X_T - X_A$$
$$= \frac{1}{3} - \frac{19}{54}$$
$$= \frac{18-19}{54}$$
$$\text{Error} = \frac{-1}{54}$$

$$\text{Relative error} = \frac{\text{Error}}{\text{True value}}$$

$$\boxed{\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}}$$

$$= \frac{-1/54}{1/3}$$
$$= -\frac{1}{18}$$

$$\text{Percentage error} = (\text{Relative error}) \times 100$$
$$= \frac{-1}{18} \times 100$$
$$= \frac{-100}{18} \%$$

Note:

1) In Trapezoidal rul we fit linear curve in each subinterval so if the actual curve is also linear then trapezoidal rule will give no error in approximation of integral over those curves.

2) If by Trapezoidal rule we find approximate integral of a curve in interval $[x_0, x_n]$ in which curve is convex then approximate value of the integral will be greater than true value of the integral.

Q. If we approximate $\int_0^1 x^4 \, dx$ by trapezoidal rule of steplength 0.01 then approximate value of the integral will be —— ?

1) $1/5$ 2) $> 1/5$ 3) $< 1/5$ 4) None

$\Rightarrow$

$f(x) = x^4$

$f''(x) = 12x^2 > 0$ in $(0,1)$

$\therefore$ $f(x)$ is convex in $(0,1)$

$\therefore$ Approx. Value $>$ True Value

$\therefore$ Approx. Value $> \frac{1}{5}$

Ans : (2)

Q. If we want to find approximate value of integral $\int_0^1 (3x+7) \, dx$ by trapezoidal rule of the steplength 0.0001 then it's value will be —— ?

1) $3/2$ 2) 7 3) $17/2$ 4) None

$\Rightarrow$ By Note (1) above,

The given curve is linear then the Trapezoidal rule will give no error.

$\therefore$ Approx Value = Actual value

$= \frac{17}{2}$

$\therefore$ Ans = (3).

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2) Midpoint Approximation.

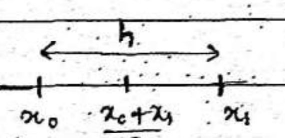
If we have set of tabulated points (x_i, y_i), $i=0,1,\dots,n$ then $\int_{x_0}^{x_n} y\,dx$ is approximated in each subinterval by the functional value at the midpoint

$$\int_{x_0}^{x_n} y\,dx \approx \sum_{i=0}^{n} h f(m_i) \quad \text{Where for } [x_{i-1}, x_i] \quad m_i = \frac{x_i + x_{i-1}}{2}$$

Error in Approximation:

$$X = \int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots$$

$$X_{Approx} = h f\left(\frac{x_0 + x_1}{2}\right)$$



$$= h f\left(x_0 + \frac{h}{2}\right)$$

$$= h\left\{y_0 + \frac{h}{2}y_0' + \frac{(h/2)^2}{2!}y_0'' + \cdots\right\}$$

$$X_{True} = h y_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0''' + \cdots$$

$$\text{Error } (E) = X_{True} - X_{Approx}$$

$$= h^3\left(\frac{1}{6} - \frac{1}{8}\right)y_0'' + \cdots$$

$$= \frac{h^3}{24}y_0'' + \cdots$$

$\therefore$ Error in $[x_0, x_1] = \frac{h^3}{24}y_0''$

$\therefore$ Total error,

$$E = \frac{h^3}{24}\left[y_0'' + y_1'' + y_2'' + \cdots + y_{n-1}''\right]$$

$y_i)$

e

$, x_j]$

$+ x_{j-1}$
2

$$|E_{[a,b]}| \leq \frac{h^3}{24} |f''(x)|^{max}$$

$$\leq \frac{(b-a)}{12} h^2 |f''(x)|^{max}.$$

Q. If $\int^1 f(x)\,dx$ is approximated by $\sum_{j=1}^{n} f(m_j).h$ where m_i is the midpoint of i^{th} subinterval in partition of $[0,1]$ in partition of $[0,1]$, then modulus or magnitude of error in approximation is bounded by ___ ?

$$\Rightarrow \int_0^1 f(x)\,dx \approx \sum_{j=1}^{n} f(m_i).h$$

$$|E| \leq \frac{(b-a)}{24} h^2 |f''(x)|^{max}$$

$$\leq \frac{h^2}{24} |f''(x)|^{max}, \text{ on } [0,1].$$

3) Simpsons $(1/3)^{rd}$ Rule :

If in approximation of integration we take three tabulated points in each subintegral then this process of approximation results into simpsons $(1/3)^{rd}$ rule.

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \cdots + \int_{x_{n-2}}^{x_n} y\,dx \quad (1)$$

In $[x_0, x_2]$

$$\int_{x_0}^{x_2} y\,dx \approx h\int_0^2 \left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0\right)dp$$

$$\approx h\left[2y_0 + \frac{p^2}{2}\Delta y_0\Big|_0^2 + \cdots\right]$$

$$\approx h\left[2y_0 + 2\Delta y_0 + \frac{(8/3 - 2)}{2!}\Delta^2 y_0\right]$$

$$\approx h\left[2y_0 + 2(y_1 - y_0) + (1/3)(y_2 - 2y_1 + y_0)\right]$$

$$\approx \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

eq^n (1) becomes

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\right]$$

Note:

Simpsons $(1/3)^{rd}$ rule will be applicable over set of datas which are multiple of three, multiple of 2+1, ..

$$\int_{x_0}^{x_n} y\,dx = 3 + 2 + 2 + \ldots$$

Multiple of 2 + 1 = M(2) + 1

Q. By taking steplength 1/4, find the approximate value of $\int_0^1 x^2 dx$ by simpsons $(1/3)^{rd}$ rule also find an error.

⇒

x	0	1/4	2/4	3/4	4/4 = 1
y	0	1/16	1/4	9/16	1

twice

Thrice

once

$$\int_0^1 x^2 dx \approx \frac{1}{12}\left[(0+1)+2\left(\frac{4}{16}\right)+4\left(\frac{1}{16}+\frac{9}{16}\right)\right]$$

$$\approx \frac{1}{3}$$

As simpsons $(1/3)^{rd}$ rule takes three tabulated points in each subintegral so if the actual curve is quadratic then approximate value and true value of the integral will coincide.

Q. By simpsons $(1/3)^{rd}$ rule the approximate value of $\int_0^1 (5x^2+6x+7)\,dx$ by taking steplength of 1/50 will be

1) 5/3 2) 3 3) 7 4) 35/3

$$\Rightarrow \int_0^1 (5x^2+6x+7)\,dx = \frac{5}{3}+3+7$$

$$= \frac{35}{3}$$

Actual value of $\int_{x_0}^{x_2} y\,dx$

$$= \int_{x_0}^{x_2} \left(y_0 + (x-x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0^{iv} + \cdots \right) dx \quad (4)$$

$$= 2h y_0 + \frac{(2h)^2}{2!} y_0' + \frac{(2h)^3}{6} y_0'' + \frac{(2h)^4}{24} y_0''' + \frac{(2h)^5}{120} y_0^{iv}$$

Approximate value in $[x_0, x_n]$

$$\approx \frac{h}{3} \left[y_0 + 4y_1 + y_2 \right]$$

$$\approx \frac{h}{3} \left[y_0 + 4\left(y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots \right) + \left(y_0 + 2h y_0' + \frac{(2h)^2}{2!} y_0'' + \cdots \right) \right]$$

Error in $[x_0, x_2] = X - X_A$

$$= h^5 \left(\frac{32}{120} - \frac{5}{18} \right) y_0^{iv}$$

$$= \frac{-h^5}{90} y_0^{iv}.$$

Total error $E = \frac{-h^5}{90} \left[\underbrace{y_0^{iv} + y_2^{iv} + \cdots + y_{n-2}^{iv}}_{n/2 \text{ terms}} \right]$

$$= \frac{-h^5}{90} \times \left(\frac{n}{2} \right) \tilde{y}^{iv}$$

$$= -\frac{(b-a)}{90} h^4 \tilde{y}^{iv}$$

Where $\tilde{y}^{iv} = \dfrac{y_0^{iv} + \cdots + y_{n-2}^{iv}}{n/2}$

Which is the mean value of the 4^{th} derivative at starting point of subintervals.

(4) Simpsons $(3/8)^{th}$ rule

In this method of approximating integral we take four tabulated points in each subintegral,

Hence $\displaystyle\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_3} y\,dx + \int_{x_3}^{x_6} y\,dx + \cdots + \int_{x_{n-3}}^{x_n} y\,dx$

And it is approximated by,

$$\int_{x_0}^{x_3} y\,dx \approx h\int_0^3 y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots$$

$$\approx h\left[3y_0 + \frac{9}{2}\Delta y_0 + \cdots\right]$$

$$\therefore \int_{x_0}^{x_n} y\,dx \approx \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

$$+ \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$\vdots$$

$$+ \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

$$\approx \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + \cdots + y_{n-2} + y_{n-1})\right.$$

$$\left. + 2(y_3 + y_6 + \cdots + y_{n-3}) + y_n\right]$$

Note

Simpsons $(3/8)^{th}$ rule is applicable if tabulated points are of the form $M(3)+1$

~~i.e. 4, 7, 10, 13, ...~~

In the actual curve for approximation of integral by simpsons $(3/8)^{th}$ rule is of order upto 3 then approximate value and the value of the integral will coincide.

Q. By simpsons $(3/8)^{th}$ rule approximate value of integral $\int_0^{10}(x^3+x^2-x+1)\,dx$ by taking steplength $(1/10)$ will be ___ ?

$\Rightarrow$ 1) $\frac{10^4}{4}+\frac{10^3}{3}+\frac{10^2}{2}+10$

2) 10

3) 0

4) None

As set of points by steplength $1/10$ are

$$\underbrace{0 \quad 1/10 \quad 2/10 \quad 3/10 \ \cdots \ 100/10}_{\cancel{101}\text{ points}}$$

$101 \neq M(3)+1$

as $101 = 99+2$

$= M(3)+2 \neq M(3)+1$

$\therefore$ This formula not applicable

Ans : (4).

M(3)+1

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10 are

Q. $\int_0^{10}(x^3+x^2-x+1)\,dx$, steplength 10/6

$\Rightarrow$ $\underbrace{0 \quad 10/6 \quad 20/6 \quad 30/6 \quad 40/6 \quad 50/6 \quad 60/6}_{7 \text{ points}}$

$7 = 6+1$

$= M(3)+1$

$\therefore$ formula is applicable and the value is Ans : (1) above

Error

Actual value (X) :

$$X = \int_{x_0}^{x_3} y\,dx = \int_{x_0}^{x_3} y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \cdots$$

$$= 3hy_0 + \frac{(3h)^2}{2!}\,y_0' + \frac{(3h)^3}{3!}\,y_0'' + \cdots$$

Approximate value: in $[x_0, x_n]$

$$X_A \approx \int_{x_0}^{x_3} y\,dx = \frac{3h}{8}\left[y_0+3y_1+3y_2+y_3\right]$$

X_A in $[x_0, x_n]$ is

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[y_0+3y_1+3y_2+y_3\right]$$

$$+\frac{3h}{8}\left[y_3+3y_4+3y_5+y_6\right]$$

$$+ \vdots$$

$$\approx \frac{3h}{8}\left[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$$

$$\approx \frac{3h}{8}\left[y_0+3(y_1+y_2+\cdots y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})+y_n\right]$$

- Numarical Sol^{n}s of ODE -

If we have function of two variables x & y then $f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$

$$+ h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + \cdots$$

is known as Taylors expansion of $f(x,y)$.
In this topic we will be given $y' = f(x,y)$ and initial condition $y(0) = y_0$ i.e $y(x_0) = y_0$.
In this topic we have two goals:

1) Evaluate $y(x)$.
2) Evaluate $y(x_1) = y_1$

1) Taylors method (Not in syllabus).
2) Picards method (In syllabus).
3) Eulers method.
4) Improved Euler.
5) Modified Euler.
6) Runge kutta method.

1) Taylors Method: (Not in syllabus)

$$y(x) = y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \cdots$$

$y(0) = y_0$

$y_0' = f(x_0, y_0)$

$y'' = f_1(x, y, y')$

$y_0'' = f_1(x_0, y_0, y_0')$

$\vdots$

$y^n = F_{n-1}(x, y, y', y'', \cdots, y^{n-1})$

2) Picards Method.

$$y' = \frac{dy}{dx} = f(x,y) \text{ -- given}$$

$$dy = f(x,y)\,dx$$

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\therefore\ y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

1st approximation:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x,y^{(1)})\,dx$$

preceeding in this mannar,

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x,y^{n-1})\,dx$$

$$n = 1,2,\dots,\quad y^{(0)} = y(0)$$

Q. Consider $y' = x + y$. and $y(0) = 1$
find y as a function of x
1) By Taylors method upto 3rd power x
2) By picards method upto 2nd iteration.

$\Rightarrow \quad y' = x + y \quad , \quad y'(0) = y(0) = 1$

$y'' = \cancel{y}1 + y' \quad , \quad y''(0) = 1 + y'(0) = 2$

$y''' = y'' \qquad y'''(0) = 2$

$$y = 1 + (x - x_0) \cdot 1 + \frac{(x-x_0)^2}{2!}(2) + \frac{(x-x_0)^3}{3!}(2)$$

$$= 1 + (x - 0) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(2)$$

$$= 1 + x + x^2 + \frac{x^3}{3}.$$

By picards method:

$$y^{(1)} = 1 + \int_0^x f(x,y)\,dx$$

$$= 1 + \int_0^x (x+1)\,dx$$

$$= 1 + \frac{x^2}{2} + x$$

$$y^{(2)} = 1 + \int_0^x \left(\frac{x^2}{2} + x + 1\right) dx$$

$$= 1 + \frac{x^3}{6} + \frac{x^2}{2} + x$$

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3. Eulers Method.

To find values of y at certain points

$$\left.\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y(0) = y_0 \end{aligned}\right\} \text{given}$$

$$y(x_1) = ?$$

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y_1 - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

In Eulers method to find approximate value of $y(x)$

let $f(x, y) = f(x_0, y_0)$ in $[x_0, x_1]$

$$y_1 \approx y_0 + h f(x_0, y_0)$$

4. Improved Eulers method. (Explicit trapezoidal method)

Insted of approximating the value of $f(x, y)$ at initial point $f(x, y)$ was taken as mean of the value at the two end points.

$$f(x, y) = \frac{f(x_0, y_0) + f(x_0, \tilde{y}_1)}{2}$$

where $\tilde{y}_1 = y_0 + h f(x_0, y_0)$ ... by Eulers above

$$y_1 \approx y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, \tilde{y}_1)\right]$$

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5. <u>Modified Eulers Method.</u>

In this method n^{th} approximation at y_1 was taken as by trapezoidal rule

$$y_1^{(n)} = y_0 + \int_{x_0}^{x_1} \left(\frac{f(x_0, y_0) + f(x_1, y_1^{n-1})}{2} \right) dx$$

$$y_1^{(n)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{n-1}) \right]$$

$$n = 1, 2, 3, \ldots$$

$$y_1^{(0)} = y_0 + h f(x_0, y_0).$$

Q Consider differential equation.
$y' = x + y$ subject to the condition $y(0) = 1$ find value of $y(0.1)$ by taking steplength 0.1.

1) By Eulers method.
2) By Improve Euler method.
3) Modified Euler.

$\Rightarrow$ $f(x, y) = x + y$

$f(x_0, y_0) = 0 + 1 = 1$

$h = 0.1$

1) $y(0.1) = y(0) + h f(x_0, y_0)$
$= 1 + (0.1) \cdot 1$
$= 1.1$

2) $y(0.1) = y(0) + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, \tilde{y}_1) \right]$

$h = 0.1, \quad \tilde{y}_1 = 1.1$

$$f(x_1, \tilde{y}_1) = 0.1 + 1.1$$
$$= 1.2$$

$$\therefore y(0.1) = 1 + \frac{0.1}{2}[1 + 1.2]$$
$$= 1.11$$

(3) $y_1^{(n)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{n-1})\right]$

$$y_1^{(n)} = 1 + \frac{0.1}{2}\left[1 + 0.1 + y_1^{n-1}\right]$$
$$= 1 + 0.05\left[1.1 + y_1^{(n-1)}\right]$$

$y_1^{(0)} = 1.1$ by Eulers method.

$$y_1^{(1)} = 1 + 0.05[1.1 + 1.1]$$
$$= 1 + 0.05(2.2)$$
$$= 1.11$$

$$y_1^{(2)} = 1 + 0.05[1.11 + 1.11]$$
$$= 1.1105$$

All compair all the three approximate value with the actual value.
Actual sol? of

$$(D-1)y = x \qquad y' = x + y$$

is $y(x) = 2e^{x} - (x+1)$

$$y(0.1) = 2e^{0.1} - (1.1).$$

6

Error in Eulers method.

$$E = 2e^{0.1} - 1.1 - 1.1$$
$$= 2e^{0.1} - 2.2$$

6. Runge - Kutta method. (R-K method)

In this method of finding approximate value of y_1, Error were taken into consideration and according to which different order R-K methods were developed.

If we have R-K method of order k then error will be of the order h^{k+1}

The superhit R-K method is R-K method of order 2 that means error will be of order 3.

R-K method of order 2 is given by

$$y_1 = y_0 + W_1 k_1 + W_2 k_2$$

where $k_1 = hf(x_0, y_0) = hf_0$

$$k_2 = hf(x_0 + \alpha h, y_0 + \beta k_1)$$

As $y' = f(x, y)$, $y(0) = y_0$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}$$

$$y_0 + W_1(hf_0) + W_2\left[hf(x_0, y_0) + \alpha h \frac{\partial f}{\partial x} + \beta h f_0 \frac{\partial f}{\partial y}\right]$$

$hf_0 \qquad W_1 + W_2 = 1$

$h^2 \frac{\partial f}{\partial x} \qquad W_2 \alpha = \frac{1}{2}$

$h^2 \frac{\partial f}{\partial y} \qquad W_2 \beta = \frac{1}{2}$

$$\alpha = \beta = 1$$

$$\Rightarrow W_1 = \frac{1}{2}, \quad W_2 = \frac{1}{2}$$

and $k_1 = hf_0$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$$\therefore y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

Q. By Range-Kutta method
find 1) K_1
2) K_2
3) $y(0.1)$
for initial value problem
$y' = x + y, \; y(0) = 1.$

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$\Rightarrow$ $K_1 = hf(x_0, y_0)$

$= (0.1) \times [0+1]$

$= 0.1$

$K_2 = hf(x_0+h, y_0+k_1)$

$= 0.1(0.1+1.1)$

$= 0.12$

$y(0.1) = 1 + \frac{h}{2}(0.1+0.12)$

$= 1.11$